17th International Symposium on Ice
Saint Petersburg, Russia, 21 - 25 June 2004
International Association of Hydraulic Engineering and Research

Session B

MATHEMATICAL MODELLING OF PROCESS OF ICE-SNOW COVER FORMING IN RESERVOIR

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ABSTRACT

1D 3-layered model describing ice cover growth in the water body with slow-moving water is developed. This model takes into account snow accumulation on ice cover surface. The emphasis was placed on description of the heat transfer process in three conjugated mediums: water, ice and snow. The "small parameter" problem was solved while numerical realization of the model. The numerical simulations basing on meteorological data for the Novosibirsk reservoir was carried out. The good congruence of simulations results with the field measurement data was displayed.

MATHAMATICAL STATEMENT OF THE PROBLEM

The mathematical statement comes to solution of the heat conductivity equations in three conjugated areas with three desired moving boundaries: "water-ice" ($z = f_1(t)$), "ice-snow" ($z = f_2(t)$), "snow-atmosphere" ($z = f_3(t)$) (fig.1).
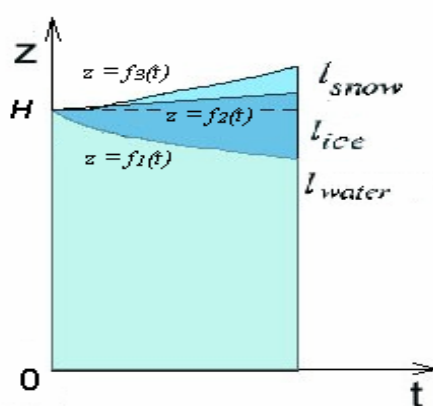

Fig. 1. Definitional domain of the problem

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BASIC EQUATIONS

$$0 < z < f_1(t): \quad \frac{\partial T_{water}}{\partial t} = a_{water}^2 \frac{\partial^2 T_{water}}{\partial z^2}; \tag{1}$$

$$f_1(t) < z < f_2(t): \quad \frac{\partial T_{ice}}{\partial t} = a_{ice}^2 \frac{\partial^2 T_{ice}}{\partial z^2}; \tag{2}$$

$$f_2(t) < z < f_3(t): \quad c_p \rho(z) \frac{\partial T_{snow}}{\partial t} = \frac{\partial}{\partial z}\left(k_{snow} \frac{\partial T_{snow}}{\partial z} \right), \tag{3}$$

where T is the temperature of the respective medium, °C; a^2 is the thermal diffusivity, m^2/s; $k_{snow}=2.85 \cdot 10^{-6}\rho^2(z)$ (a − $Abels$ $G.P.$ [6]) or $k_{snow}=2.9 \cdot 10^{-6}\rho^2(z)+0.043$ (b − [5]) is the thermal conductivity of snow, W/m°C; $\rho_{snow}(z) = \rho_0 e^{b(f_3(t)-z)}$ is the density of snow, kg/m³, b − const; ρ_0 is the density of freshly fallen snow; λ is the latent heat of ice fusion, J/kg; c_p is the specific heat of snow, J/kg °C.

Edge conditions:
at the bottom of a reservoir:

$$\frac{\partial T_{water}}{\partial z}\bigg|_{z=0} = 0 \text{ or } \quad T_{water}\big|_{z=0} = T_{bottom},$$

where T$_{bottom}$ is the temperature of water at the bottom of a reservoir; at the "water-ice" interface $z = f_1(t)$ is defined Stefan's condition:

$$k_{water}\frac{\partial T_{water}}{\partial z}\bigg|_{z=f_1(t)} - k_{ice}\frac{\partial T_{ice}}{\partial z}\bigg|_{z=f_1(t)} = \lambda\rho_{ice}\frac{dl_{ice}}{dt}, \tag{4}$$

$$T_{water}\big|_{z=f_1(t)} = T_{ice}\big|_{z=f_1(t)} = 0^0\text{C};$$

where l_{ice} is the thickness of ice cover; at the "ice-snow" interface $z = f_2(t)$:

$$k_{ice}\frac{\partial T_{ice}}{\partial z}\bigg|_{z=f_2(t)} = k_{snow}\frac{\partial T_{snow}}{\partial z}\bigg|_{z=f_2(t)}, \tag{5}$$

$$T_{ice}\big|_{z=f_2(t)} = T_{snow}\big|_{z=f_2(t)},$$

at the "snow-air" interface $z = f_3(t)$:

$$T_{snow}\big|_{z=f_3(t)} = T_a(t),$$

where T_a is the air temperature.

As initial conditions at $t = 0$ the positions of all moving boundaries and vertical heat distribution are specified.

Some problems appear while simulating the initial stage of ice cover forming, because movement velocity of the "water-ice" interface is theoretically equal to infinity in the initial time of the process. This problem was overcome by introduction of the squared ice cover thickness as a unknown quantity $S(t)=l^2_{ice}(t)$

$$\frac{dl_{ice}}{dt} = \frac{d\sqrt{S}}{dt} = \frac{dS}{2l_{ice}dt} .$$

METHOD OF SOLUTION
The front rectification method, allowing the numerical solution of the mentioned equations set in the regular domain is used (fig.2) [2, 3].

Changing over to the new variables:

$\bar{t} = t, \quad 0 < \xi_i < 1(i = 1,2,3)$

$\xi_i = \dfrac{z - f_{i-1}(t)}{f_i(t) - f_{i-1}(t)}.$

Fig.2. Transformation of the definitional domain

The positions of moving boundaries is defined by the change of phase equations and mass conservation law

$$f_1 (t) = l_{water} = H - K_\rho\, l_{ice} , \; K_\rho =\rho_{ice}/\rho_{water};$$
$$f_2 (t) =f_1(t) + l_{ice}= H + (1 - K_\rho)l_{ice}(t);$$
$$f_3 (t) = f_2 (t)+ l_{snow} ;$$

$l_{snow} = \dfrac{\ln(1 + bl^*(t))}{b}$ is the depth of snow cower taking into account the shrinkage of snow;

$$l^* = 0.001W \frac{\rho_{water}}{\rho_0} ,$$

where $l^*(t)$ is the thickness of freshly fallen snow (m) with density ρ_0; W is a water equivalent of snow, mm; H – is a reservoir depth, m.

Basic equations in the new variables
$$l_{ice}l^2_{water} \frac{\partial T_{water}}{\partial \bar{t}} = a^2_{water}l_{ice} \frac{\partial^2 T_{water}}{\partial \xi^2} - \upsilon_{water} \frac{\partial T_{water}}{\partial \xi}; \quad \upsilon_{water} = \frac{1}{2} K_\rho l_{water}\xi \frac{dS}{d\bar{t}}; \qquad (1')$$

$$S\frac{\partial T_{ice}}{\partial \bar{t}} = a_{ice}^2 \frac{\partial^2 T_{ice}}{\partial \xi^2} - \upsilon_{ice}\frac{\partial T_{ice}}{\partial \xi}; \quad \upsilon_{ice} = \frac{1}{2}(K_\rho - \xi)\frac{dS}{d\bar{t}}; \qquad (2')$$

$$l_{ice}l_{snow}^2 \frac{\partial T_{snow}}{\partial \bar{t}} = a_{snow}^2 (\rho)\left[-2bl_{snow}\frac{\partial T_{snow}}{\partial \xi} + \frac{\partial^2 T_{snow}}{\partial \xi^2}\right] - \upsilon_{snow}\frac{\partial T_{snow}}{\partial \xi};$$

$$\upsilon_{snow} = \frac{1}{2}(K_\rho - 1)l_{snow}\frac{dS}{d\bar{t}} - l_{snow}l_{ice}\xi\frac{dl_{snow}}{d\bar{t}}, \qquad (3')$$

$$a_{snow}^2 = \frac{k_{snow}}{\rho c_p}.$$

Coupling conditions in the new variables

$$\frac{k_{water}l_{ice}}{l_{water}\lambda\rho}\frac{\partial T_{water}}{\partial \xi}\bigg|_{\xi_1=1} - \frac{2k_{ice}}{\lambda\rho}\frac{\partial T_{ice}}{\partial \xi}\bigg|_{\xi_2=0} = \frac{dS}{d\bar{t}},$$

$$T_{water}\big|_{\xi_1=1} = T_{ice}\big|_{\xi_2=0}; \qquad (4')$$

$$\frac{k_{ice}}{l_{ice}}\frac{\partial T_{ice}}{\partial \xi}\bigg|_{\xi_2=1} = \frac{k_{snow}}{l_{snow}}\frac{\partial T_{snow}}{\partial \xi}\bigg|_{\xi_3=0},$$

$$T_{ice}\big|_{\xi_2=1} = T_{snow}\big|_{\xi_3=0}. \qquad (5')$$

ALGORITHM OF NUMERICAL SOLUTION
The implicit directed-difference scheme was used for approximation of basic equations 3-point sweep method was used for the solution of the obtained system of finite-difference equations. The simulations were performed with 1 day time step at the uniform grid with 10 points in each area.

ANALYTICAL SOLUTION
If temperature distribution through ice thickness is taken as linear while simulation of the ice cover thickness without taking into account snow cover (because the processes of heat transfer through ice are quasistationary) and the thermal flux from water body is neglected, then we obtain the equation for the determination of ice thickness:

$$\lambda\rho_{ice}\frac{dl_{ice}}{dt} = -k_{ice}\frac{T_a(t)}{l_{ice}}, \quad \text{from which we obtain} \quad l_{ice}(t) = \sqrt{\frac{2k_{ice}}{\lambda\rho_{ice}}\int_0^t (-T_a(\tau))d\tau} \quad \text{under}$$

$l_{ice}(0) = 0$, which is according with the results given in [1] and the results obtained by empirical formula given in [4].

NUMERICAL RESUALTS
The numerical simulations of ice cover growth dynamics in the Novosibirsk reservoir under meteorological data of Ordynskoe meteorological station (middle part of reservoir) and Obskaya hydro-meteorological observatory (lower part of reservoir) were carried out. The date with steady average daily air temperature below -7°C and wind speed below *5 m/s* is used. It accords to the conditions of freezing-over beginning [5].

The results of numerical simulation and analytical solution of the problem not taking into account snow cover as far as field measurement are given in fig. 3. Here the meteorological data of Obskaya hydro-meteorological observatory since October 31 1989 and until January 31 1990 are used. The analytical and numerical solutions are in close agreement. Of course, the calculated values of ice cover thickness are bigger than the measured ones because the influence of snow cover is not taken into account.
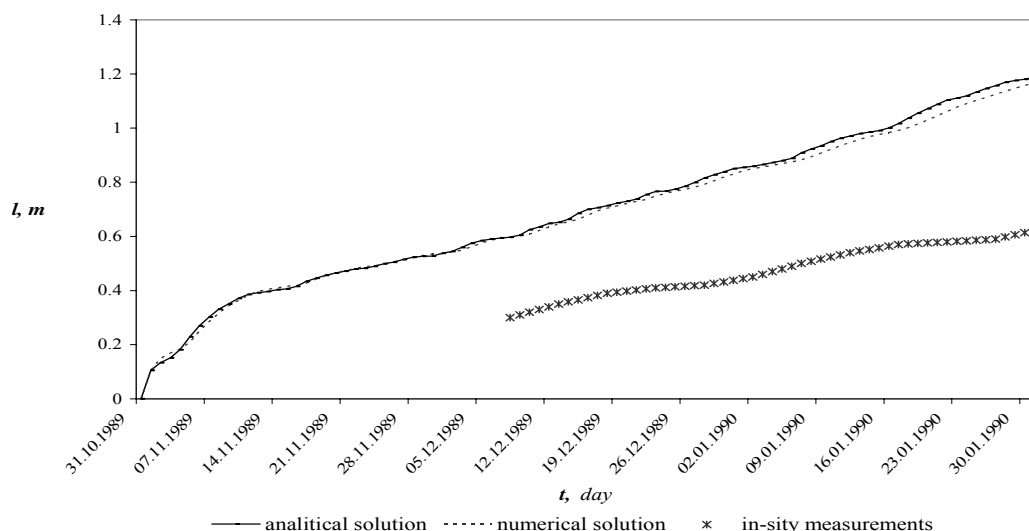

Fig.3. The ice cover thickness growth dynamics without taking snow cover into account

The comparison of ice cover thickness obtained by the offered model using different formulas for thermal conductivity of snow with the ice cover thickness obtained by empirical formulas of Piotrovich (1968), method of Devic [6] and with the field measurement data is given in fig. 4.
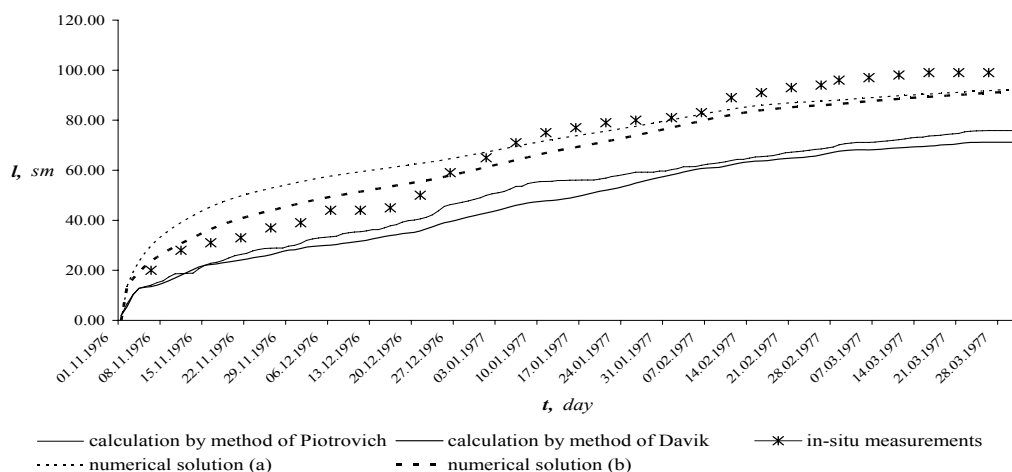

Fig.4. The dynamics of ice cover thickness growth in Novosibirsk reservoir under meteorological data for years 1976-1977 for Ordynskoe meteorological station
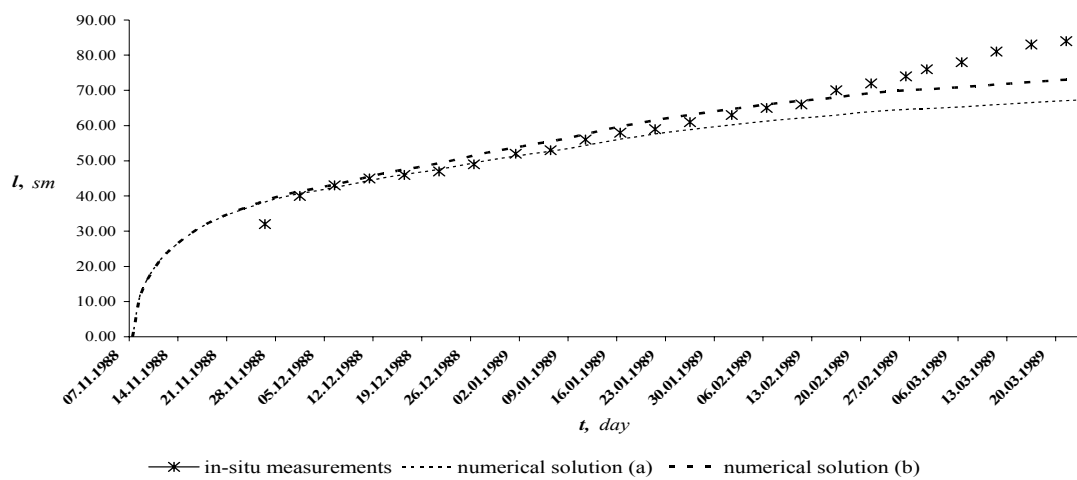
Fig.5. The dynamics of ice cover thickness growth in Novosibirsk reservoir under meteorological data for years 1988-1989 for Ordynskoe meteorological station (the results of numerical simulations in comparison with the field measurement data)
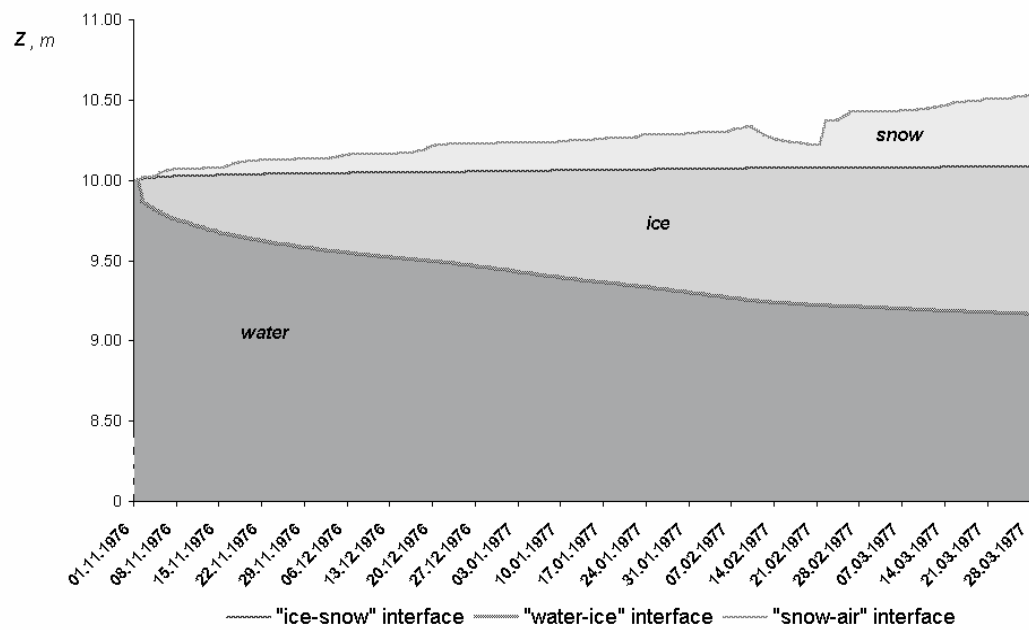

Fig.6. The movement of boundaries of layers while the numerical simulation of ice cover thickness under meteorological data for years 1976-1977 for Ordynskoe meteorological station
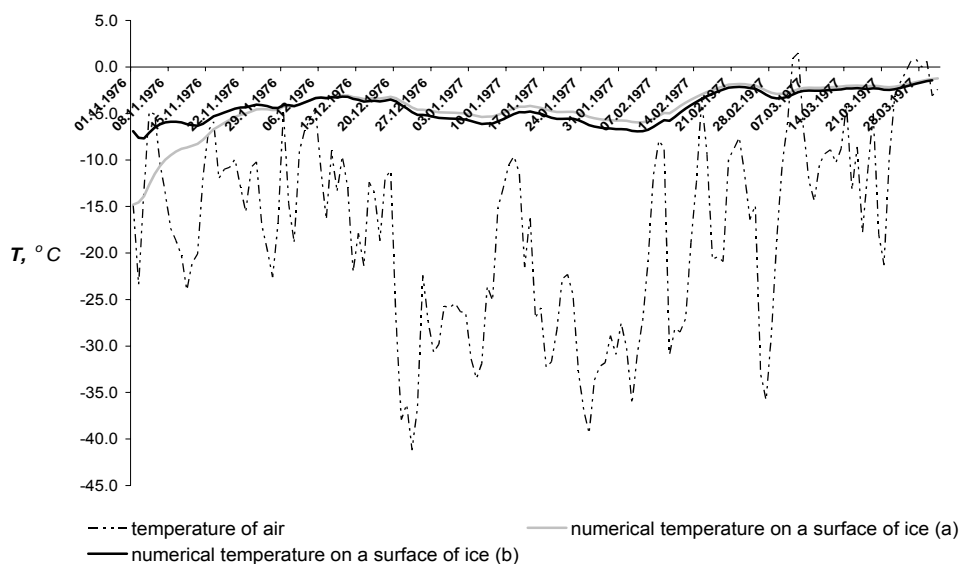
Fig.7. The dynamics of temperature at the ice-snow boundary while the numerical simulation of ice cover thickness under meteorological data for years 1976-1977 for Ordynskoe meteorological station

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